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A COMPARISON OF INCREMENTAL BEHAVIOUR OF ELASTOPLASTIC AND CLoE MODELS

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SUMMARY

A comparison between two classes of constitutive equations is presented. The comparison is based on two models, namely the classical bi-linear elastoplastic Von Mises model and an incrementally non linear model previously proposed by the authors (called CLoE model). Both models exhibit the same behaviour for isotropic and deviatoric loading paths. Gudehus' diagram, inversibility, second order work and shear band localization criteria are comparatively analysed. As a result the two constitutive equations share some similarities but the CLoE model gives a more appropriate and flexible description of shear band formation. Copyright © 1999 John Wiley & Sons, Ltd.

Key words: constitutive equations; Von Mises model; CLoE model; plasticity

1. INTRODUCTION

Many constitutive equations for solid materials have been developed in the past 20 years. The most frequent approach to model solid behaviour is the flow theory of plasticity. In this framework, the simplest models are based on a single plastic mechanism. However, it was earlier seen, in order to remedy to some extent the discrepancy between theoretical results and experimental evidence of bifurcation and for micromechanical reasons as well, that realistic modeling needs at least a multimechanism constitutive equation. The limit case of such models is a thoroughly incrementally non linear law. Earlier, Hill[1,2] proposing a general framework for such a model, restricted himself to models admitting a velocity gradient potential. He gave results about the uniqueness of the rate problem for this class of models. Moreover, Petryk[3,4] has extended the results available for these models to instability and loss of ellipticity. The main assumption which allows all these results is the existence of the velocity gradient potential. This assumption implies (but the reverse is false) that the corresponding continua is standard.

It is well-known that geomaterials (soils, rocks, or concrete) do not obey the normality rule, they are not standard materials. However, it is known that for such materials too, a vertex model is necessary for a good modeling[5]. The classical results which come essentially from the so-called comparison solid theory[6] do not hold. A complete (which means study of continuous as well as discontinuous bifurcation in the Rice sense[7]) shear band study is possible but not straightforward[8];

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moreover as pointed out by Petryk[9], frequently the constitutive relations at the vertex are not fully specified.

Using a thoroughly incrementally non-linear model is an alternative way which may seem less simple. However, recently[10,11], we proposed a framework of incremental non-linear models called CLoE. The two main features of which are embedded in the name. First, the concepts of limit surface and of flow rule for perfectly plastic states are linked in a consistency condition (which explains the 'C') which can be seen as a weak form (here only available for perfect plastic states) of the corresponding one in classical elastoplastic models. Second, although the model is thoroughly non-linear a localization analysis can be done explicitly (which explain the 'Lo' and the 'E'). This framework was used to develop a complete constitutive equation for soils[12]. In order to understand these models better we want, in this paper, to investigate some aspects of a very simple case by comparison with the classical corresponding elastoplastic model. In order to be tractable, the comparison is restricted here to Von Mises materials but the corresponding CLoE model does not admit a velocity gradient potential.

Four points of view are considered to compare between the classical isotropic hardening elastic–plastic models and CLoE models, namely

1. the Gudehus' diagrams[13];
2. the second-order work;
3. the localization criterion in the classical Rice's sense[7];
4. and the inversibility of the constitutive equations (which yields to what is often called search of limit points).

For the sake of simplicity, the study is restricted to infinitesimal strains, so there is no distinction between the different stress tensors and between the different stress rates.

2. PRELIMINARIES

2.1. Notation

In the sequel, $\boldsymbol{\sigma}$ denotes the stress tensor; $\boldsymbol{\varepsilon}$ the strain tensor; $\dot{\boldsymbol{\sigma}}$ the stress rate; $\dot{\boldsymbol{\varepsilon}}$ the strain rate; $\mathbf{s}$ the stress deviator; $\dot{\mathbf{s}}$ the stress rate deviator; $\mathbf{e}$ the strain deviator; $\dot{\mathbf{e}}$ the strain rate deviator; $\mathbf{i}$ is the second-order identity tensor; $\mathbf{L}$ the fourth-order identity tensor. $\mathbf{E}$ is the elasticity isotropic tensor which can be defined by

$$\mathbf{E} = K(\mathbf{i} \otimes \mathbf{i}) + 2G(\mathbf{L} - \tfrac{1}{3}\mathbf{i} \otimes \mathbf{i}) \tag{1}$$

where K is the bulk modulus and G the shear modulus.

Five deviatoric second-order tensors written in the stress principal axes such as

$$\mathbf{j} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\dfrac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{k} = \begin{bmatrix} \sqrt{\dfrac{2}{3}} & 0 & 0 \\ 0 & -\dfrac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\dfrac{1}{\sqrt{6}} \end{bmatrix}$$

$$\mathbf{l}=\begin{bmatrix}0 & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\qquad \mathbf{m}=\begin{bmatrix}0 & 0 & \frac{1}{\sqrt{2}}\\ 0 & 0 & 0\\ \frac{1}{\sqrt{2}} & 0 & 0\end{bmatrix},\qquad \mathbf{n}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & 0\end{bmatrix}$$

are useful in the following. These tensors are five independent eigenvectors of $\mathbf{E}$ corresponding to the same eigenvalue $2G$. $\mathbf{j}$ and $\mathbf{k}$ have the same principal axes as the stress. Moreover, the norm of these tensors is equal to 1. These tensors and the tensor $(1/\sqrt{3})\mathbf{i}$ (corresponding to the eigenvalue $3K$ are the six elements of an orthonormal basis for the second-order tensors. Let us define

1. $\mathbf{p}=(\mathbf{s}/\|\mathbf{s}\|)$ where $\mathbf{s}$ is deviatoric stress tensor;
2. $\mathbf{q}$ such that $\|\mathbf{q}\|=1$ and $\mathbf{p}:\mathbf{q}=0$ and has the same principal axes as $\mathbf{p}$ (or $\mathbf{s}$);
3. δ such that $\mathbf{p}=\cos\delta\mathbf{j}+\sin\delta\mathbf{k}$ and $\mathbf{q}=-\sin\delta\mathbf{j}+\cos\delta\mathbf{k}$.

It is worth noticing that $\mathbf{p}$ and $\mathbf{q}$ are orthogonal to $\mathbf{l}$, $\mathbf{m}$ and $\mathbf{n}$. Any unit second-order tensor such as a strain rate $\dot{\boldsymbol{\varepsilon}}$ can be written as a linear combination of the six tensors $(1/\sqrt{3})\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$: $\dot{\boldsymbol{\varepsilon}}=b_1\mathbf{j}+b_2\mathbf{k}+a_3\mathbf{l}+a_4\mathbf{m}+a_5\mathbf{n}+a_6(1/\sqrt{3})\mathbf{i}$ where b_1, b_2, a_3, a_4, a_5 and a_6 are scalars meeting the constraint: $b_1^2+b_2^2+a_3^2+a_4^2+a_5^2+a_6^2=1$, alternatively such unit strain rates can be written as

$$\dot{\boldsymbol{\varepsilon}}=a_1\mathbf{q}+a_2\mathbf{p}+a_3\mathbf{l}+a_4\mathbf{m}+a_5\mathbf{n}+a_6(1/\sqrt{3})\,\mathbf{i} \tag{2}$$

where a_1, a_2, a_3, a_4, a_5 and a_6 meet the constraint

$$a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2=1 \tag{3}$$

It will be interesting to study three particular cases for deviatoric strain rate, which allows us to define α, β and γ:

$$\dot{\boldsymbol{\varepsilon}}=\dot{\mathbf{e}}=\cos\alpha\mathbf{q}+\sin\alpha\mathbf{p}$$

$$\dot{\boldsymbol{\varepsilon}}=\dot{\mathbf{e}}=\cos\beta\mathbf{l}+\sin\beta\mathbf{p}$$

$$\dot{\boldsymbol{\varepsilon}}=\dot{\mathbf{e}}=\cos\gamma\mathbf{l}+\sin\gamma\mathbf{q}$$

As said previously it is possible to define for the CLoE model a limit surface and a flow rule. In this paper we restrict our study to Von Mises models. This means that the limit surface for studied CLoE models and the yield surface for elastoplastic models are the well-known Von Mises surfaces defined by

$$\psi=\sqrt{\mathbf{s}:\mathbf{s}}-c=0$$

where c is a value which depends on the history for the elastoplastic model. The flow rule which can be defined in both cases (see Section 4.1 for CLoE) is the same and corresponds to a flow rule normal to the Von Mises surface. This means that both models are associative but the studied CLoE model does not derive from a velocity gradient potential as it will be seen in Section 5.

2.2. Gudehus' diagrams

Given a state, a model is fully described by a mapping between the strain rate space and the stress rate space. Restricting to rate-independent models, it is sufficient to study the mapping

between an hypersphere of unit radius centred at the origin of the strain rate space and the corresponding hypersurface in the stress rate space. These diagrams, hereafter called Gudehus' diagrams, show clearly some of the properties of the models. However, a part of the properties of the models can remain hidden in these diagrams, because only cutaways of the hypersurface are drawn.

2.3. Second-order work

In the following, given a state, we study the value of

$$\Delta W = \dot{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} \tag{4}$$

for every $\dot{\boldsymbol{\varepsilon}}$. As the studied models are rate independent this study has to be done only for unit strain rate.

The second-order work value is often interpreted as a property of the tangent moduli tensor. For incrementally non-linear models the tangent tensor is stress direction (or strain direction) dependent[14,15] and thus an interpretation in terms of a tangent tensor has no intrinsic meaning.

For classical elastoplastic models, it is well-known[16] that if the second-order work is positive for every strain rate, uniqueness is proved for some boundary value problems. For such models, the positiveness of the second order work is thus a sufficient condition of uniqueness. For boundary value problems involving CLoE models, a sufficient condition for existence and uniqueness of the solution has been established and this sufficient condition implies that the second-order work is positive[17]. Moreover, now it is proved that the positiveness of the second-order work itself implies existence and uniqueness of the corresponding boundary value problem involving CLoE models[18].

3. ELASTOPLASTIC VON MISES MODELS

The simplest form of Von Mises elastic–plastic constitutive equations can be summarized as follows.

The normal to the yield surface associated with the yield function ψ:

$$\mathbf{f} = \frac{\partial \psi}{\partial \boldsymbol{\sigma}} = \mathbf{s}$$

defines the direction of the plastic strain rate. If h is the strain hardening modulus, then the elastic plastic constitutive equation reads.

If $\psi = 0$ and $\mathbf{f} : \mathsf{E} : \dot{\boldsymbol{\varepsilon}} = 2G\mathbf{s} : \dot{\boldsymbol{\varepsilon}} > 0$ then plastic loading occurs and

$$\dot{\boldsymbol{\sigma}} = \mathsf{D} : \dot{\boldsymbol{\varepsilon}}$$

with

$$\mathsf{D} = \mathsf{E} - \frac{1}{a}\mathsf{E} : \mathbf{f} \otimes \mathbf{f} : \mathsf{E} = \mathsf{E} - \frac{4G^2}{h + 2G}\frac{\mathbf{s} \otimes \mathbf{s}}{\mathbf{s} : \mathbf{s}} = \mathsf{E} - \frac{4G^2}{h + 2G}\mathbf{p} \otimes \mathbf{p}$$

and

$$a = h + \mathbf{f} : \mathsf{E} : \mathbf{f} = \mathrm{h} + 2\mathrm{G}$$

2. else

$$\dot{\boldsymbol{\sigma}} = \mathbf{E} : \dot{\boldsymbol{\varepsilon}}$$

This model has been extensively studied in the past and the following results are well-known. However graphical illustrations (Figures 1–3), allow us to more deeply understand the model. Moreover, this is also useful to compare with the corresponding diagrams of CLoE models.

(a) The model is inversible provided $h > 0$.
(b) The second-order work is positive if and only if $h > 0$
For any unit strain rate written like in equation (2) (see also equation (3)) we can write:
(i) if the response is purely elastic (i.e. $a_2 < 0$)

$$\dot{\boldsymbol{\sigma}} = 2G(a_1\mathbf{q} + a_2\mathbf{p} + a_3\mathbf{l} + a_4\mathbf{m} + a_5\mathbf{n}) + 3Ka_6\frac{1}{\sqrt{3}}\mathbf{i} \tag{5}$$

The second-order work (see equation (4)) reads

$$\Delta W = 3Ka_6^2 + 2G(1 - a_6^2) = 2G + (3K - 2G)a_6^2$$

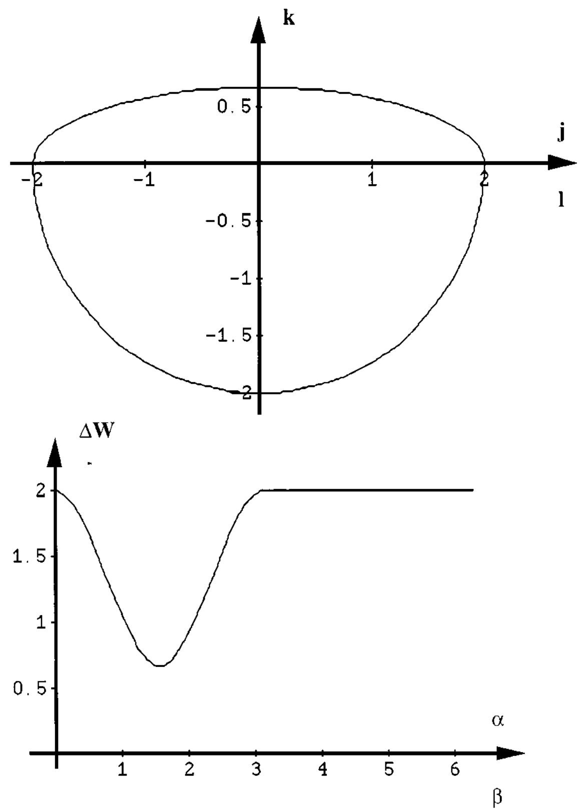

Figure 1. Gudehus diagram and variation of the second-order work for a Von Mises elastoplastic model ($h = 1$, $G = 1$)

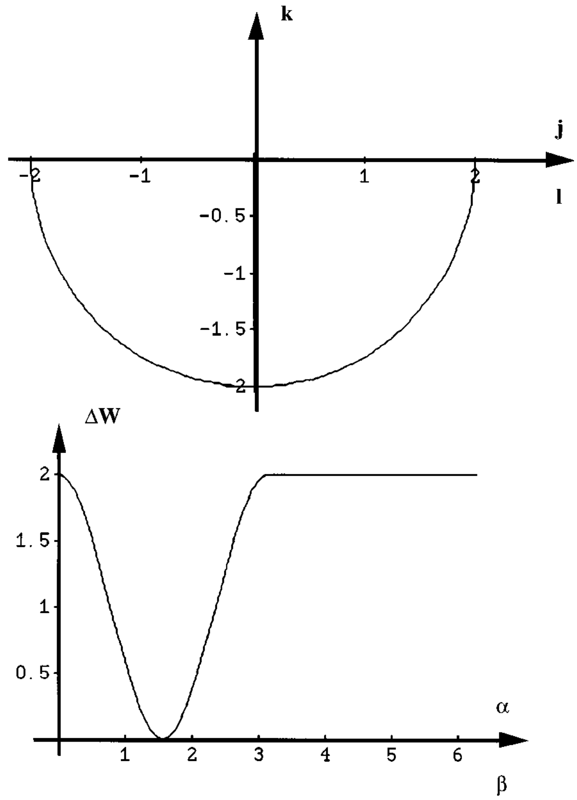

Figure 2. Gudehus diagram and variation of the second-order work for a Von Mises elastoplastic model ($h = 0$, $G = 1$)

if $a_6^2 = 0$ this yields

$$\Delta W = 2G$$

(ii) If the response is elastoplastic (i.e. $a_2 > 0$)

$$\dot{\boldsymbol{\sigma}} = 2G(a_1\mathbf{q} + a_3\mathbf{l} + a_4\mathbf{m} + a_5\mathbf{n}) + \frac{2Gh}{2G+h}a_2\mathbf{p} + 3Ka_6\frac{1}{\sqrt{3}}\mathbf{i} \tag{6}$$

The second-order work reads

$$\Delta W = (3K - 2G)a_6^2 + 2G\frac{2G(1-a_2^2)+h}{2G+h}$$

if $a_6^2 = 0$, this yields

$$\Delta W = 2G\frac{2G(1-a_2^2)+h}{2G+h}$$

As $a_2^2 < 1$ (see equation (3)) it is clear that the second-order work can be negative if and only if $h < 0$.

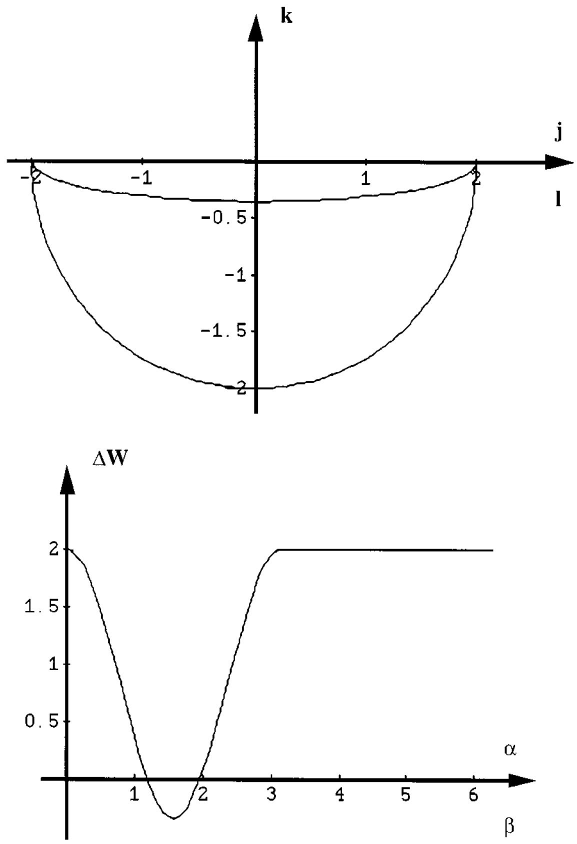

Figure 3. Gudehus diagram and variation of the second-order work for a Von Mises elastoplastic model ($h = -0{\cdot}3$, $G = 1$)

(c) Shear band localization is impossible if $h > 0$. If $h = 0$ it is only possible for stress states such as $\lambda \mathbf{j}$ (where λ is any scalar) or tensor obtained by circular permutation of the axes. For other states a negative value for hardening modulus is necessary, the critical value of which is given by the formulae

$$h_{\text{crit}} = -\tfrac{2}{3} E \sin^2 \delta$$

where E is the Young modulus.†

In the following, the model is studied for two different stress states: $\mathbf{s} = \lambda \mathbf{k}$ ($\mathbf{p} = \mathbf{k}$) and $\mathbf{s} = \lambda \mathbf{j}$ ($\mathbf{p} = \mathbf{j}$) and for three values of the hardening parameter: $h = 1$, $h = 0$ and $h = -0{\cdot}3$. Examples are computed with $G = 1$.

† This result is only apparently different from the one of Ottosen and Runesson[19] as we use here a different but equivalent yield function

(d) Figure 1 shows the Gudehus' diagrams for $h = 1$, and for $\dot{\boldsymbol{\varepsilon}} = \dot{\mathbf{e}} = \cos\alpha\mathbf{q} + \sin\alpha\mathbf{p} = \cos\alpha\mathbf{j} + \sin\alpha\mathbf{k}$ or for $\dot{\boldsymbol{\varepsilon}} = \dot{\mathbf{e}} = \cos\beta\mathbf{l} + \sin\beta\mathbf{p} = \cos\beta\mathbf{l} + \sin\beta\mathbf{k}$ and the variation of the second order work as a function of α (or β). As $\mathbf{m}$ and $\mathbf{n}$ are as $\mathbf{l}$ orthogonal to $\mathbf{p}$ and $\mathbf{q}$ and then to the plastic strain it is not necessary to study other strain rates to well understand the whole behaviour associated with this elastoplastic model. Figure 2 shows the corresponding Gudehus' diagrams and the variation of the second order work for $h = 0$. Figure 3 shows the corresponding diagrams and the variation of the second order order work for $h = -0{\cdot}3$. If $\dot{\boldsymbol{\varepsilon}} = \dot{\mathbf{e}} = \cos\gamma\mathbf{l} + \sin\gamma\mathbf{q}$ the Gudehus' diagrams are in every case centred circles and the second-order work is constant and equal to $2G$.

(e) The Gulehus' diagrams in the case of $\mathbf{s} = \lambda\mathbf{j}$, $\mathbf{p} = \mathbf{j}$ are not drawn as they are similar to the previous ones with with interchange between $\mathbf{k}$ and $\mathbf{j}$. The variations of the second order work are the same as the previous ones if we perform interchanges between β and γ and between α and $\alpha + \pi/2$.

The previous results explain why localization is possible with $h = 0$ for the second case ($\mathbf{s} = \lambda\mathbf{j}$) but not in the first one ($\mathbf{s} = \lambda\mathbf{k}$). Localization is possible as soon as second order work becomes negative or equal to zero for a strain rate which can be written as $\frac{1}{2}(\boldsymbol{g} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{g})$ i.e. for a strain rate for which at least one principal value equals zero and the other two have different signs. In the first case the second-order work becomes zero for $h = 0$ and for a strain rate equal to $\mathbf{k}$ which does not meet the condition. The set of strain rates for which the second order work is negative has to be large enough to include a tensor which meets the previous condition (one principal value equal to zero, the other two having different signs). Consequently, the value of $-h$ has to be large enough to allow localization. Conversely, in the second case the second-order work becomes first negative for a tensor which corresponds to shear banding and localization is possible as soon as $h = 0$.

4. CLoE CONSTITUTIVE EQUATIONS

CLoe models are built on a unique but non-linear relation between the stress rate and the strain rate[20,10,12]. A very simple non-linear relation is assumed

$$\dot{\boldsymbol{\sigma}} = \mathbf{A}:(\dot{\boldsymbol{\varepsilon}} + \mathbf{B}\|\dot{\boldsymbol{\varepsilon}}\|) = \mathbf{A}:\dot{\boldsymbol{\varepsilon}} + \mathbf{b}\|\dot{\boldsymbol{\varepsilon}}\| \tag{7}$$

$\mathbf{A}$ is a fourth-order tensor, and $\mathbf{B}$ and $\mathbf{b}$ are second-order tensors such that $\mathbf{A}:\mathbf{B} = \mathbf{b}$. These tensors depend on the assumed state variables of the material under consideration.

4.1. Limit surface and flow rule

If $\|\mathbf{B}\| = 1$ and $\dot{\boldsymbol{\varepsilon}} = -\lambda\mathbf{B}$, λ being any positive real number it is clear from equation (7) that the corresponding stress rate $\dot{\boldsymbol{\sigma}}$ is zero. If we accept the definition of the limit surface as the set of stress states for which a strain rate can exist with vanishing stress rate then $\|\mathbf{B}\| = 1$ is the equation of the limit surface and $-\mathbf{B}$ is the flow rule. If $\mathbf{A}$ is inversible it is easy to prove that if $\|\mathbf{B}\| < 1$ the constitutive equation is inversible, which means that it is possible to express $\dot{\boldsymbol{\varepsilon}}$ as a function of $\dot{\boldsymbol{\sigma}}$. Conversely, if $\|\mathbf{B}\| \geqslant 1$, the constitutive equation is no more inversible. Let us denote in the following $\psi = \|\mathbf{B}\| - 1$ in order to compare CLoE and classic elastic–plastic constitutive equations.

4.2. Consistency condition

If no stress rate can be directed outside the limit surface defined by $\psi = 0$, it is possible to prove[10–12] that for states lying on the limit surface:

$$\frac{\partial \psi}{\partial \boldsymbol{\sigma}} : \mathbf{A} = -\lambda \mathbf{B}$$

with λ a positive real number. There is an important difference between this condition and the consistency condition in classical elasto-plasticity, in the present case the condition does not concern a yield surface but only the limit surface. Hence it has to be met only in an asymptotic manner, which is an advantage for computational algorithms.

4.3. Comments

This class of models has some similarity with endochronic theory and this paper has more or less the same objective as the one of Bazant[14]: a comparison between the classical elastic–plastic models with an alternative way of modelling elastoplastic behaviour. However, there is not in CLoE models any decomposition of the strain rate and no part of the model is called elastic. The model of Kolymbas[21] now known as the hypoplastic model can be written in the same manner. A significant difference between CLoE and the hypoplastic models is the consistency condition (not considered in Kolymbas model).

4.4. Shear band analysis and Rice's criterion

Rice[7] studied the possibility of a bifurcation involving a shear band into a previously homogeneous media. The difference of the velocity gradient inside and outside the shear band is assumed to be equal to: $\boldsymbol{g} \otimes \boldsymbol{n}$, where $\boldsymbol{n}$ is the vector normal to the shear band and $\boldsymbol{g}$ any vector. Equilibrium equation in the rate form are written and then the constitutive equation is used. Whatever the model, this analysis cannot be seen exactly as a condition of non-uniqueness for a boundary value problem as some boundary conditions are not necessarily met[22]. An approach of the latter point may be seen in the work of Benallal[23].

When the constitutive equation is incrementally linear it turns out that shear bands are possible when the determinant of the acoustic tensor corresponding to the normal $\boldsymbol{n}$ vanishes. This interpretation of the shear band analysis is no more possible for non-linear models. If the non-linearity is a bilinearity (it is the case for instance for elastic plastic models associative or not) it can be proved by different ways that it is sufficient to study only the linear problem with the loading behaviour[24,8,19].

For the CLoE model it is possible to do the localization analysis analytically despite its incremental non-linearity. It can be proved[20,25,26] that shear bands are first possible when for at least one $\boldsymbol{n}$,

$$\left\| \tfrac{1}{2}\left(\mathbf{P}_{il}^{-1}\mathbf{b}_{lk}\boldsymbol{n}_k\boldsymbol{n}_j + \mathbf{P}_{jl}^{-1}\mathbf{b}_{lk}\boldsymbol{n}_k\boldsymbol{n}_i\right)\right\| = 1 \tag{8}$$

where the summation convention is assumed, $\mathbf{P}^{-1}$ is the inverse of $\mathbf{P}$ and

$$\mathbf{P}_{ij} = \mathbf{A}_{ikjl}\boldsymbol{n}_k\boldsymbol{n}_l \tag{9}$$

can be seen as the acoustic tensor related to $\mathbf{A}$.

This localization criterion can be seen in the following manner:[26] There exists some vector $\boldsymbol{g}$ and an associated vector $\boldsymbol{n}$ such that if $\dot{\boldsymbol{\sigma}}$ is the stress rate given by the constitutive equation for $\dot{\boldsymbol{\varepsilon}}_{kl} = \frac{1}{2}(\boldsymbol{g}_k \boldsymbol{n}_l + \boldsymbol{g}_l \boldsymbol{n}_k)$ then $\dot{\boldsymbol{\sigma}}_{ij}\boldsymbol{n}_j = 0$ and

$$\|\tfrac{1}{2}(\boldsymbol{g}_k \boldsymbol{n}_l + \boldsymbol{g}_l \boldsymbol{n}_k)\| = 1 \tag{10}$$

This implies

$$(\tfrac{1}{2}\mathbf{A}_{ijkl}(\boldsymbol{g}_k \boldsymbol{n}_l + \boldsymbol{g}_l \boldsymbol{n}_k) + \mathbf{b}_{ij})\boldsymbol{n}_j = 0 \tag{11}$$

If equation (11) is solved in $\boldsymbol{g}$ and the solution put into equation (10), we get immediately the criterion given by equation (8).

The satisfaction of localization criterion implies that the second-order work becomes negative[26]. This result which is classical for linear models is still true for CLoE models and has the same consequences. The larger the set of strain rates for which the second-order work is negative, the more likely is shear band localization.

5. VON MISES CLoE MODEL

We choose

$$\mathbf{A} = \mathbf{E} \tag{12}$$

and

$$\mathbf{B} = -\frac{2G}{2G + h}\mathbf{p} = -\Gamma\mathbf{p} \tag{13}$$

In the following we will use

$$\Gamma = \frac{2G}{2G + h} \tag{14}$$

We can then verify that the normality holds, and that the consistency condition (Section 4.2) is met. However, this model does not admit a velocity gradient potential. Linearizing the model[27] yields $\dot{\boldsymbol{\sigma}} = \mathbf{C}:\dot{\boldsymbol{\varepsilon}}$ with $\mathbf{C} = \mathbf{E} - \mathbf{E}:\Gamma\mathbf{p}:\otimes(\dot{\boldsymbol{\varepsilon}}/\|\dot{\boldsymbol{\varepsilon}}\|)$ which is not in general symmetric. With these assumptions, for any unit strain rate defined as in equation (2) and meeting condition (3), we have

$$\dot{\boldsymbol{\sigma}} = 2G(a_1\mathbf{q} + a_2\mathbf{p} + a_3\mathbf{l} + a_4\mathbf{m} + a_5\mathbf{n} - \Gamma\mathbf{p}) + 3Ka_6\frac{1}{\sqrt{3}}\mathbf{i} \tag{15}$$

The meaning of h is the following. With the previous formulae, for a strain rate such that $\dot{\boldsymbol{\varepsilon}} = \dot{\mathbf{e}} = \lambda\mathbf{s}$ ($\lambda > 0$) which means $a_2 = 1$, $a_1 = a_3 = a_4 = a_5 = a_6 = 0$, and for the same value of h, this model and the classical elastoplastic Von Mises model (see Section 3) give the same stress rate:

$$\dot{\boldsymbol{\sigma}} = \dot{\mathbf{s}} = \Gamma h\mathbf{p} + 3Ka_6\frac{1}{\sqrt{3}}\mathbf{i} \tag{16}$$

So if h is chosen in the same manner in the two models (as a function of state), the two models predict the same behaviour for the isotropic stress path (clasical elastic behaviour means h is infinite in the two cases) and for deviatoric radial stress paths ($\dot{\mathbf{s}} = \lambda\mathbf{s}$). We can thus call K and G as elastic parameters in the sense that for $s = 0$ the behaviour predicted by the model is reversible and only governed by the fourth-order tensor $\mathbf{A} = \mathbf{E}$.

5.1. Inversibility

We have (see Eq. (13))

$$\|\mathbf{B}\| = |\Gamma|$$

and so the model is inversible provided that $h > 0$. If $h = 0$ plastic flow occurs, and the flow rule is $\mathbf{p}$ (which is the same result as for the classical Von Mises elastoplastic model).

5.2. Study of the second-order work

From equations (2) and (15), we get

$$\Delta W = (3K - 2G)a_6^2 + 2G(1 - \Gamma a_2)$$

Let us compute the minimum of ΔW.

1. If $3K - 2G > 0$ it is clear that in order to minimize ΔW, we have to minimize a_6 and to maximize a_2. So the minimum value for ΔW corresponds to $a_2 = 1$ and $a_6 = 0$. Then the second-order work reads

$$\Delta W = 2G(1 - \Gamma)$$

   It is clear that $h > 0$ implies $\Delta W > 0$ (see equation (14)), $h < 0$ is a necessary condition for the negativeness of the second-order work.
2. If $3K - 2G < 0$ we have to maximize a_6^2 and a_2 which are linked by the relation: $a_6^2 + a_2^2 \leqslant 1$ (see equation (3)). So we are sure to minimize ΔW with the constraints $a_6^2 = 1 - a_2^2$. In this case,

$$\Delta W = 3K(1 - a_2^2) + 2G(a_2^2 - \Gamma a_2)$$

   the minimum of which can be negative for $h \geqslant 0$. But $3K - 2G < 0$ means that the Poisson ratio is less than 0 or greater than 0·5 which is completely unrealistic.

Finally, for reasonable values of the elastic parameters the same results about the sign of the second-order work are obtained for classical elastoplastic model and for this Von Mises CLoE model 1.

5.3. Shear band localization

As recalled in Section 4.4, shear band localization is possible for CLoE models only if the second order work can have a null value. The previous results has then the consequence that shear band bifurcation is impossible as long as $h > 0$. But as the model studied here is quite simple (although non-linear) we can get more detailed results. As seen in Section 4.4 we search a vector $\boldsymbol{g}$ and an associated unit vector $\boldsymbol{n}$ such that equations (11) and (10) hold.

Substituting equations (1) and (12) and inverting equation (11) yields

$$\boldsymbol{g}_i = \frac{1}{G}\left((-\mathbf{b}_{ij})\boldsymbol{n}_j + \frac{3K + G}{3K + 4G}\boldsymbol{n}_l \mathbf{b}_{lk}\boldsymbol{n}_k \boldsymbol{n}_i\right) \tag{17}$$

The linear equation (11) is now solved in $\boldsymbol{g}$: the solution put into the non-linear equation

$$\tfrac{1}{2}(\boldsymbol{g}_k\boldsymbol{g}_k + (\boldsymbol{g}_i\boldsymbol{n}_i)^2) = 1$$

which is equivalent to equation (10). It gives the localization criterion

$$\frac{1}{2G^2}(\boldsymbol{n}_l\mathbf{b}_{lk}\mathbf{b}_{ki}\boldsymbol{n}_i + \Lambda(\boldsymbol{n}_l\mathbf{b}_{lk}\boldsymbol{n}_k)^2) = 1$$

with

$$\Lambda = \frac{2G^2 - 9K^2 - 24GK}{(3K + 4G)^2}$$

As $\mathbf{B}$ is purely deviatoric $\mathbf{b} = 2G\mathbf{B}$, using equation (13) yields the criterion

$$2(\Gamma)^2(\boldsymbol{n}_l\mathbf{p}_{lk}\mathbf{p}_{ki}\boldsymbol{n}_i + \Lambda(\boldsymbol{n}_l\mathbf{p}_{lk}\boldsymbol{n}_k)^2) = 2(\Gamma)^2F(\boldsymbol{n}) = 1$$

As for isotropic stress states h is infinite, for these states the criterion is not met. Then we have to search the maximum value of $F(\boldsymbol{n})$, for every $\mathbf{p}$ in order to compute the maximum value of h for which $2(\Gamma)^2F(\boldsymbol{n}) = 1$ (see equation (14)), which means the value of h for which shear band localization is possible. Stationary points are characterized by $\mathrm{d}F = 2\boldsymbol{n}_l(\mathbf{p}_{lk}\mathbf{p}_{ki} + \mathbf{2}\Lambda\mathbf{p}_{lk}\boldsymbol{n}_k\boldsymbol{n}_j\mathbf{p}_{ji})\,\mathrm{d}\boldsymbol{n}_i = 0$. As $\boldsymbol{n}$ is a unit vector: $\boldsymbol{n}_i\mathrm{d}\boldsymbol{n}_i = 0$ so

$$\boldsymbol{n}_l(\mathbf{p}_{lk}\mathbf{p}_{ki} + \mathbf{2}\Lambda\mathbf{p}_{lk}\boldsymbol{n}_k\boldsymbol{n}_j\mathbf{p}_{ji}) = \mu\boldsymbol{n}_i \tag{18}$$

where μ is any number. Writing equation (18) in the principal stress axes implies that at least two of the three principal values (π_1, π_2, π_3) of $\mathbf{p}$ are equal, or one component of $\boldsymbol{n}$ has a zero value.

Case 1. *Bifurcation for compression and extension axisymmetric tri-axial stress states.* Two principal values of $\mathbf{p}$ are equal, this means as $\mathbf{p}$ is deviatoric: $\mathbf{p} = \pm\,\mathbf{k}$. We can study only the case $\mathbf{p} = \mathbf{k}$ as clearly $F(\boldsymbol{n})$ does not depend on the sign of $\mathbf{p}$. Without restriction $\boldsymbol{n}_3 = 0$ can be assumed, as the problem is axisymmetrical. $\boldsymbol{n}_2^2 = 1 - \boldsymbol{n}_1^2$, thus we have

$$F(\boldsymbol{n}) = \frac{1+\Lambda}{6} + \left(\frac{1}{2} - \Lambda\right)\boldsymbol{n}_1^2 + \frac{3}{2}\Lambda\boldsymbol{n}_1^4$$

the maximum of which is attained for

$$\boldsymbol{n}_1^2 = \frac{1}{3} - \frac{1}{6\Lambda}$$

It is now easy to compute the critical value of h for which bifurcation is possible for axisymmetric states. The results are plotted as a function of the Poisson ratio ν in Figure 4 compared with the classical corresponding results for elastoplastic models. It is well known that in the latter case the critical value of the hardening modulus is $-E/6$ for instance‡). For CLoE model the order of magnitude is the same. The result depends on the value of the Poisson ratio but for any value of ν the model needs a less softening behaviour to exhibit shear band localization.

‡ This result is the same as the one of Ottosen and Runesson[19] as we use here a different but equivalent yield function

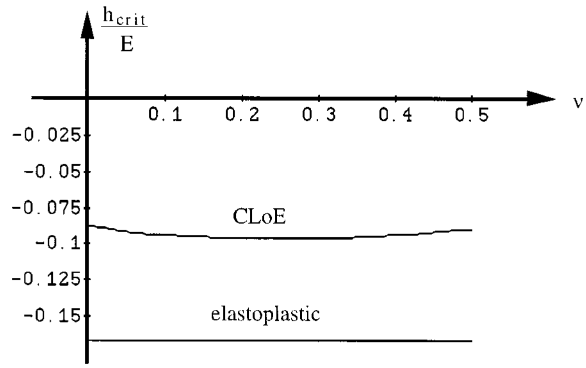

Figure 4. A comparison between the critical values of the hardening modulus for compression (or extension) stress states for a classical elastoplastic model and for the Von Mises CLoE model

Case 2. Let us denote $\boldsymbol{n}_3$ the null component $\boldsymbol{n}$. This means that the normal to the shear band is necessary inside a stress principal plane. As $\boldsymbol{n}_2^2 = 1 - \boldsymbol{n}_1^2$, we have

$$F(\boldsymbol{n}) = \boldsymbol{n}_1^2(\pi_1^2 - \pi_2^2) + \pi_2^2 + \Lambda(\boldsymbol{n}_1^2(\pi_1 - \pi_2) + \pi_2)^2$$

the maximum of which is attained for

$$\boldsymbol{n}_1^2 = \frac{\pi_1 + \pi_2 + 2\Lambda\pi_2}{2\Lambda(\pi_1 - \pi_2)}$$

For the maximum value of $F(\boldsymbol{n})$ this yields

$$-\frac{1}{4\Lambda}[(\pi_1 + \pi_2)^2 + 4\Lambda\pi_1\pi_2]$$

It can be proved that in order to maximize this result π_1 and π_2 have to be respectively the greatest and the least principal stresses. This means that in this case the normal to the shear band has to belong to the plane orthogonal to the intermediate principal direction. $\mathbf{p}$ can be written as

$$\mathbf{p} = \cos\delta\mathbf{j} + \sin\delta\mathbf{k}$$

The maximum value of $F(\boldsymbol{n})$ is then

$$\frac{-3\cos^2(\delta) + 6v^2\cos^2(\delta) - \sin^2(\delta) + 4v\sin^2(\delta) - 4v^2\sin^2(\delta)}{12v^2 - 6} \tag{19}$$

where v is the Poisson ratio. Figure 5 shows the variation of the resulting critical hardening modulus with respect to the Poisson ratio and δ. $\delta = 0$ means a pure shear stress state $\delta = \pm\pi/6$ corresponds to uniaxial stress states (compression or extension). Pure shear stress bifurcation is possible for $h = 0$. This result is the same for classical elastoplastic models. Finally, it is clear that for shear banding the two models have very similar behaviour but the CLoE model has a greater tendency to exhibit shear band localization.

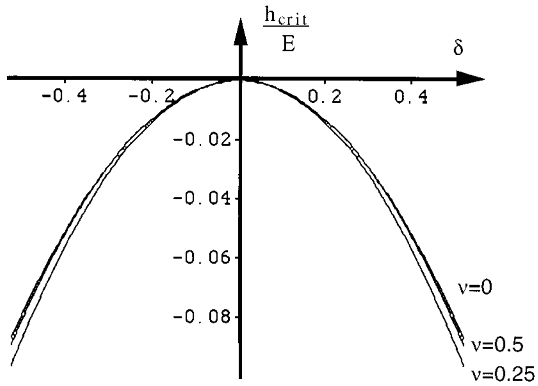

Figure 5. Critical value of the hardening modulus as a function of the Poisson ratio ν and a Lode angle δ for the Von Mises CLoE model

5.4. *Gudehus' diagrams*

Finally, Figures 6–8 represent the Gudehus' diagrams and the variations of the second order work for the present model, for $\mathbf{s} = \lambda \mathbf{k}$ and respectively for three values of the hardening modulus namely 1, 0 and $-0{\cdot}3$. In a similar manner to the classical elastoplastic model, it is possible to draw the same figure for any $\mathbf{s}$. For instance, the results are the same if the interchange between $\mathbf{k}$ and $\mathbf{j}$ is done. It is interesting to compare these results with the ones of Figures 1–3 which correspond to a classical elastoplastic model. The Gudehus' diagrams for CLoE model are rather different from the ones of elastoplastic classical models. However, about the second-order work, the results concerning the negativeness are quite similar; this explains the small differences between the critical values of the hardening parameter in the two cases. For $\dot{\boldsymbol{\varepsilon}} = \dot{\mathbf{e}} = \cos\gamma \mathbf{l} + \sin\gamma \mathbf{q}$, a projection of the Gudehus' diagram onto the $\mathbf{l}$–$\mathbf{q}$ plane gives a centred circle and the second-order work is constant and equal to $2G$.

6. ANISOTROPIC VON MISES CLoE MODEL

In the following we introduce a very slight modification which allows to adjust the bifurcation abilities of the models. Since the papers of Rudnicki and Rice[5] and Storen and Rice,[28] modification of elastoplastic models with non-coaxial terms is well-known to improve the abilities of the models to predict localization. However as recalled in the introduction in the elastoplastic framework the models are not fully specified (except the plane model of Papamichos[29] which is bi-linear). Moreover, since these models are not fully known, the corresponding localization analysis are not complete. In spite of the fact that CLoE models are non-linear, the modification done here can be seen as similar, nevertheless the model and the localization analysis are complete. CLoE models were designed with this modification in order to well fit with localization data.[11,26] Assuming that $\mathbf{A}$ is written in the stress principal axis, instead of

$$\mathbf{A}_{1212} = \mathbf{A}_{1221} = \mathbf{A}_{2112} = \mathbf{A}_{2121} = \mathbf{A}_{2323} = \mathbf{A}_{2332}$$
$$= \mathbf{A}_{3223} = \mathbf{A}_{3232} = \mathbf{A}_{3131} = \mathbf{A}_{3113} = \mathbf{A}_{1331} = \mathbf{A}_{1313} = G$$

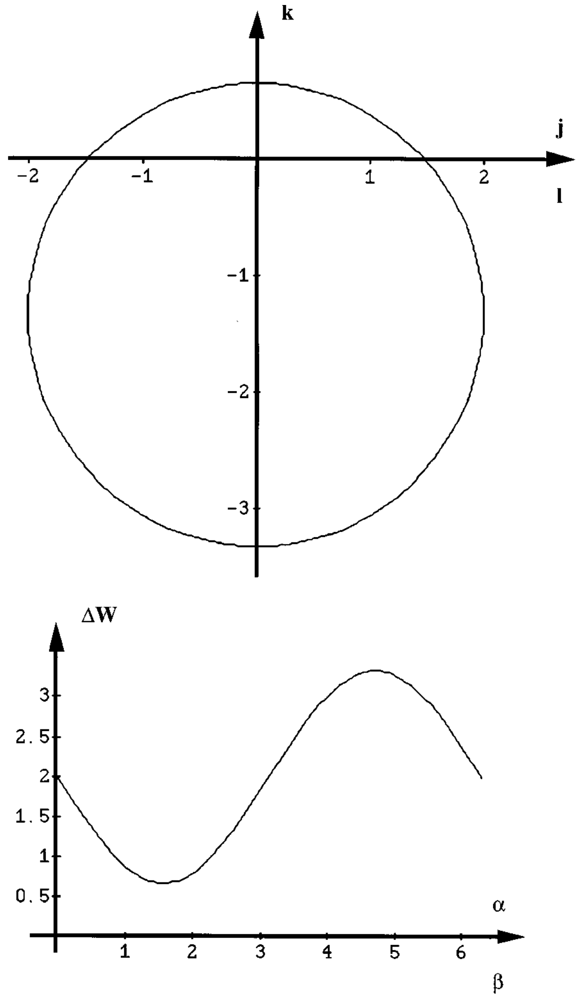

Figure 6. Gudehus diagram and variation of the second-order work for Von Mises CLoE model ($h = 1$, $G = 1$)

which is a consequence of assumptions previously done (see equations (12) and (1)) we assume

$$\mathbf{A}_{1212} = \mathbf{A}_{1221} = \mathbf{A}_{2112} = \mathbf{A}_{2121} = G_3$$

$$\mathbf{A}_{2323} = \mathbf{A}_{2332} = \mathbf{A}_{3223} = \mathbf{A}_{3232} = G_1$$

$$\mathbf{A}_{3131} = \mathbf{A}_{3113} = \mathbf{A}_{1331} = \mathbf{A}_{1313} = G_2$$

the other components of $\mathbf{A}$ remaining the same as in Section 5. This implies that $\mathbf{A}$ is anisotropic. Let us notice first that the three values have to vary with respect to the state. For instance, if the state is isotropic, $G_3 = G_1 = G_2 = G$. We do not discuss this point here. If the only state variable of the material is the stress state, a discussion has to be done.[26] In the following, it is assumed that

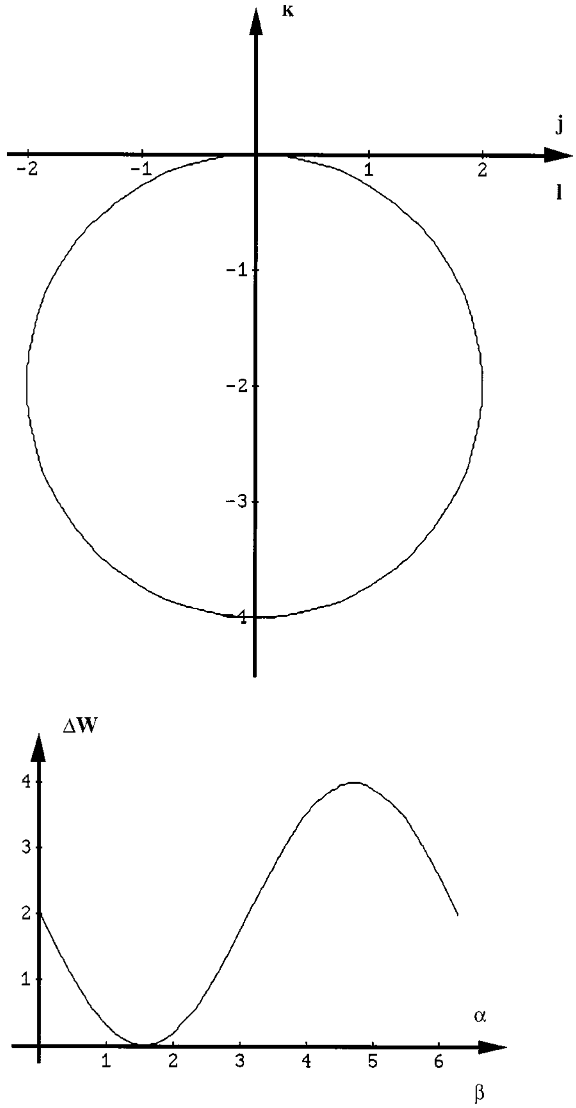

Figure 7. Gudehus diagram and variation of the second-order work for the Von Mises CLoE model ($h = 0$, $G = 1$)

the definition of the state allows us to have different values for every 'shear modulus' so that objectivity is still met. Physically, this modification can be seen as an induced anisotropy. With these assumptions the normality still holds, and the consistency condition (see Section 4) is still met. For any unit strain rate defined like equation (2) and meeting condition (3) we have

$$\dot{\boldsymbol{\sigma}} = 2G(a_1\mathbf{q} + a_2\mathbf{p} - \Gamma\mathbf{p}) + 3Ka_6\frac{1}{\sqrt{3}}\mathbf{i} + 2G_3a_3\mathbf{l} + 2G_2a_4\mathbf{m} + 2G_1a_5\mathbf{n}$$

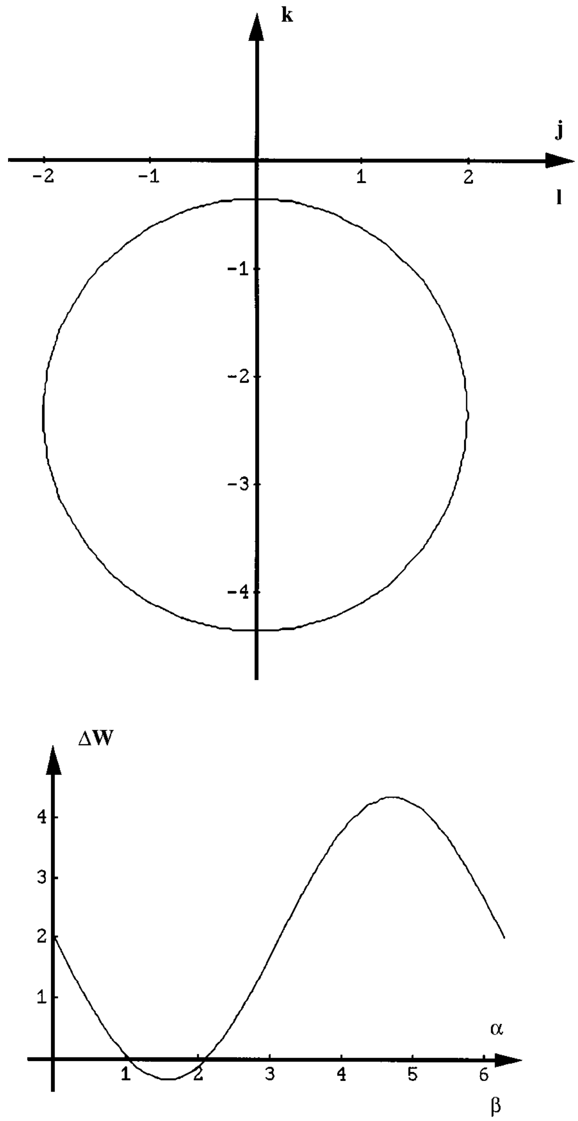

Figure 8. Gudehus diagram and variation of the second-order work for the Von Mises CLoE model ($h = -0{\cdot}3$, $G = 1$)

For loading paths for which the principal directions of the stress remain fixed, this model gives exactly the same results as the corresponding isotropic one.

6.1. Inversibility

In the following, we assume $G_3 > 0$, $G_2 > 0$, and $G_1 > 0$. So $\mathbf{A}$ is still inversible and thus (see Section 4) if h is positive, the constitutive equation is inversible.

6.2. Study of the second-order work

$$\Delta W = 3Ka_6^2 + 2G(a_1^2 + a_2^2 - \Gamma a_2) + 2G_3 a_3^2 + 2G_2 a_4^2 + 2G_1 a_5^2$$

Can the second-order work be negative? To answer we have to minimize ΔW under the constraint given by equation (3). Using the Lagrange multiplier method the minimum values of the second-order work are

$$\Delta W = 2G(a_2^2 - \Gamma a_2) + 2G_1(1 - a_2^2)$$

or the same formulae obtained by a permutation of G_1, G_2 and G_3. In the following we can then denote G_1 as the modulus which has the less value, i.e. the one for which the second order work can have its minimum value. We denote α as the ratio G_1 over G:

$$\alpha = \frac{G_1}{G}$$

(this ratio is positive as $G_1 > 0$). Thus,

$$\Delta W = 2G(a_2^2(1 - \alpha) - \Gamma a_2 + \alpha)$$

which is a quadratic function with respect to a_2. If $h > 0, \Delta W(0) > 0, \Delta W(1) > 0$. ΔW can first be negative for $a_2 = (1/2(1 - \alpha))\Gamma$ under the condition

$$0 < \frac{1}{2(1 - \alpha)}\Gamma < 1 \tag{20}$$

This yields $\Delta W = \alpha - \Gamma^2/(4(1 - \alpha))$ which is negative under condition (20) if and only if

$$\alpha < \frac{1 - \sqrt{1 - \Gamma^2}}{2}$$

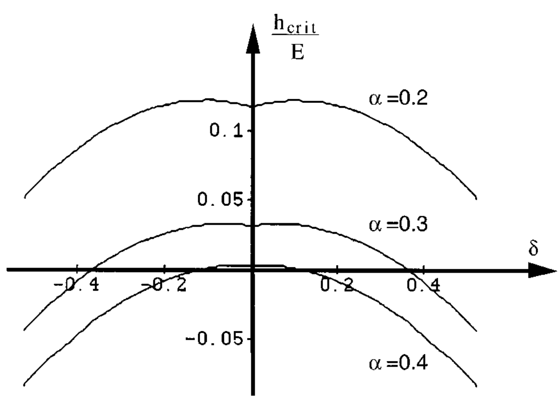

Figure 9. Critical value of the hardening modulus for strain plane localization as a function of the ratio α and the Lode angle δ for the anisotropic Von Mises CLoE model

Finally, $h < 0$ is not a necessary condition for the possible negativeness of the second-order work (see equation (14)). If $\alpha < \frac{1}{2}$ it is possible than the second-order work is negative for a value of Γ less than 1 which means for a positive value of the hardening modulus h. It is then likely that shear band localization is possible for positive values of h.

6.3. Shear band localization

Unfortunately, the shear band localization analysis is less easy to perform in this anisotropic case than in the previous isotropic one. No explicit formulae can be deduced. The shear band

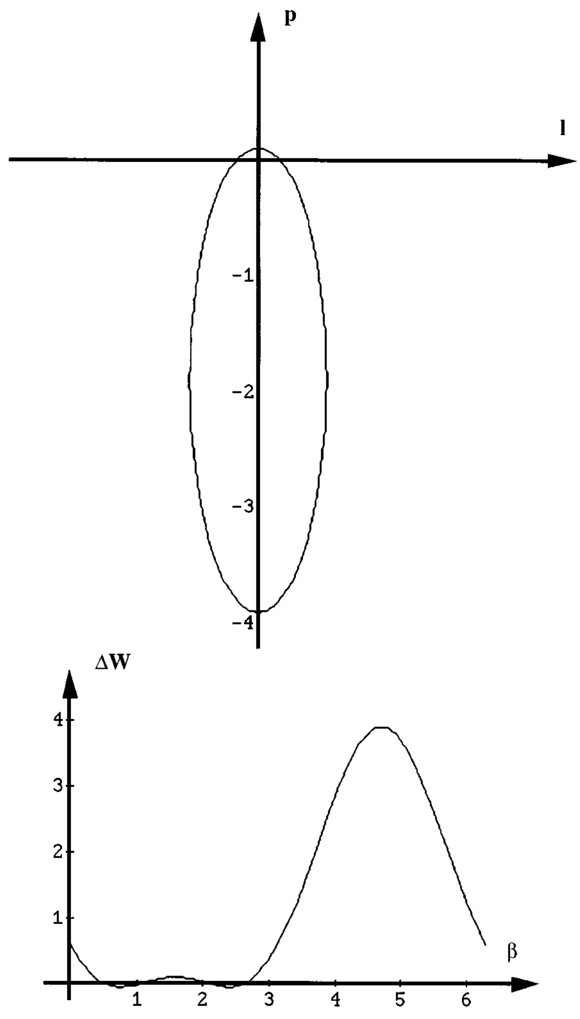

Figure 10. Gudehus diagram and variation of the second-order work for the anisotropic Von Mises CLoE model ($h = 0{\cdot}1$, $G = 1$, $\alpha = 0{\cdot}3$)

localization critierion is given by equations (8) and can be written as in Section 5.3:

$$2\Gamma^2 F(\boldsymbol{n}) - 1 = 0$$

In order to find the maximum (critical) value of h (see equation (14)) we have to find the maximum of F. A and $\mathbf{b}$ have the same symmetry planes, this implies that function F is necessary stationary for some $\boldsymbol{n}$ belonging to these symmetry planes. Unfortunately, we can have (and we numerically proved this point) stationary points which do not belong to these planes. This is due to the anisotropy, since conversely for the isotropic version of the model we proved (see Section 5) that the $\boldsymbol{n}$ corresponding to the extremum belongs to the principal plane of $\mathbf{b}$. Using anisotropic models implies to search the localization plane in the whole space.[30] In the following, we focus on a simple case. We assume that for any stress state some kinematical constraints allow only plane strain inside the plane defined by the major and minor stresses. Figure 9 shows the results for a Poisson ratio equal to 0·3 and for three values of the anisotropic parameter α namely 0·2, 0·3 and 0·4. As expected by the previous analysis about the second order work, we can see that it is possible for the model to predict shear band localization for positive hardening modulus if α is less than 0·5. Moreover, for α small enough shear banding is possible for positive hardening modulus and for any stress state.

6.4. Gudehus' diagrams

Up to now the Gudehus' diagram and the variations of the second-order work were given for $\dot{\boldsymbol{\varepsilon}} = \dot{\mathbf{e}} = \cos\alpha\mathbf{q} + \sin\alpha\mathbf{p}$ and they were similar for $\dot{\boldsymbol{\varepsilon}} = \dot{\mathbf{e}} = \cos\beta\mathbf{l} + \sin\beta\mathbf{p}$ because A was isotropic. In the present case, the Gudehus' diagrams are exactly the same as the ones of Figures 6–8, for $\dot{\boldsymbol{\varepsilon}} = \dot{\mathbf{e}} = \cos\alpha\mathbf{q} + \sin\alpha\mathbf{p}$. On the contrary, due to the anisotropy of tensor A, these diagrams are quite different from the previous ones for $\dot{\boldsymbol{\varepsilon}} = \dot{\mathbf{e}} = \cos\beta\mathbf{l} + \sin\beta\mathbf{p}$. Figure 10 shows these results. Figure 11 shows the corresponding results for $\dot{\boldsymbol{\varepsilon}} = \dot{\mathbf{e}} = \cos\gamma\mathbf{l} + \sin\gamma\mathbf{q}$. These figures especially Figure 10 show clearly the anisotropic nature of the model and concerning the second order work, the reason for which it is more likely that such a model exhibits localization than the corresponding isotropic model.

7. CONCLUSION

In this paper, non-linear CLoE models were investigated through a comparison with classical elastoplastic ones. Very often it is difficult to precisely study models as soon as they are more complex than an incrementally bilinear relation. However here, we are able to define and/or to study for these models some classical features such as limit surface, flow rule, second-order work, limit points, shear band localization. Finally, the results are very similar to classical elastoplastic ones for these models (sensitive only to the second invariant of the stress). But if we introduce some kind of anisotropy we can predict shear banding in the hardening regime. The non-linearity alone is not able to induce a significantly greater tendency to exhibit shear bands. These models are very easy to introduce in finite element codes and they are a powerful tool to study the problem of localization from a numerical point of view[27].

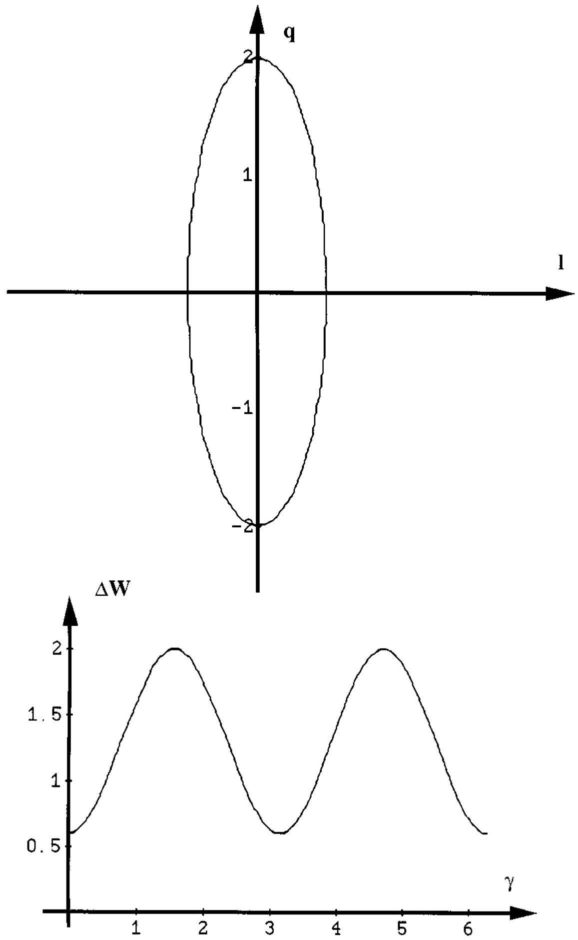

Figure 11. Gudehus diagram and variation of the second-order work for the anisotropic Von Mises CLoE model ($h = 0{\cdot}1$, $G = 1$, $\alpha = 0{\cdot}3$)

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